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LETTER TO THE EDITOR

Hamiltonian reduction and the construction of q-deformed extensions of the Virasoro algebra

E Batista†§, J F Gomes‡‖ and I J Lautenschleguer‡¶

† Universidade de São Paulo, Instituto de Física Caixa Postal 20516 01498, São Paulo, Brazil
‡ Instituto de Física Teórica-UNESP, Rua Pamplona 145, 01405-900 São Paulo, SP, Brazil

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Abstract. In this paper we employ the construction of the Dirac bracket for the remaining current of $sl(2)_q$ deformed Kac–Moody algebra when constraints similar to those connecting the $sl(2)$-Wess–Zumino–Witten model and the Liouville theory are imposed to show that it satisfies the q-Virasoro algebra proposed by Frenkel and Reshetikhin. The crucial assumption considered in our calculation is the existence of a classical Poisson bracket algebra induced in a consistent manner by the correspondence principle, mapping the quantum generators into commuting objects of classical nature preserving their algebra.

The Virasoro algebra and its extensions have been understood to provide the algebraic structure underlying conformally invariant models which includes string theory and two-dimensional (2D) statistical models on the lattice. On the other hand, quantum groups also play an important role in the integrability properties of those models (see for instance [1–4]). It thus seems natural to connect these two important subjects by constructing a q-deformed version of the Virasoro algebra and its extensions. This may prove useful in establishing a q-deformed string model in the line of [6, 7].

The construction of q-deformed Virasoro algebra have been proposed using both bosons and fermions [8]. However, a connection with the classical canonical structure is still unclear. Frenkel and Reshetikhin [5] proposed a q-Virasoro algebra based on the q-deformation of a Miura transformation involving classical Poisson brackets. The Hamiltonian reduction provides a systematic procedure in constructing extensions of the Virasoro algebra by adding to the spin 2, generators of higher spin. A typical example of such a procedure connects the Wess–Zumino–Witten (WZW) model to the 2D Toda field theories. The latter arises when a consistent set of constraints are implemented to the Kac–Moody currents describing the WZW model associated to a Lie group G [9] or to an infinite-dimensional Kac–Moody group $\hat{G}$ [10].

A redefinition of the canonical Poisson brackets into Dirac brackets is required in order for the equations of motion of the reduced model to be consistent with those obtained from the remaining current algebra. Under the Dirac bracket the spin 1 generators corresponding to the remaining currents become the W_n generators of higher spin defined according to an improved energy momentum tensor (see [9] for a review).

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For the q-deformed Kac–Moody algebras, although a canonical structure is still unknown, their algebra is well established [13, 14] and can be constructed in terms of non commuting objects (quantum fields) [11, 15].

In this paper we employ the construction of Dirac bracket for the remaining current of $sl(2)_q$ deformed Kac–Moody algebra when constraints similar to those connecting the $sl(2)$ WZW model and the Liouville theory are imposed. The crucial assumption considered in our calculation is the existence of a classical Poisson bracket algebra induced, in a consistent manner by the correspondence principle, mapping the quantum generators into commuting objects of classical nature preserving their algebra. We show that the remaining algebra coincide with the q-Virasoro algebra proposed by Frenkel [5].

For $q = 1$, the classical $sl(2)$ Poisson bracket algebra derived from the WZW model [12] is given in terms of formal power series by

$$\{H(z), H(w)\} = -\mathrm{i}k \sum_{n\in\mathbb{Z}} n \left(\frac{w}{z}\right)^n \tag{1}$$

$$\{H(z), E^{\pm}(w)\} = \mp\mathrm{i}\sqrt{2}E^{\pm}(z) \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \tag{2}$$

$$\{E^{+}(z), E^{-}(w)\} = -\mathrm{i}\sqrt{2}H(z) \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n - \mathrm{i}k \sum_{n\in\mathbb{Z}} n \left(\frac{w}{z}\right)^n . \tag{3}$$

where k characterizes the central term. The corresponding conformal Toda model associated to $G = sl(2)$ (Liouville model) is obtained by constraining [9]

$$\chi_1 = H(z) \approx 0 \qquad \chi_2 = E^{+}(z) - 1 \approx 0. \tag{4}$$

The Dirac bracket is defined by

$$\{A(z), B(w)\}_D = \{A(z), B(w)\}_P - \oint\oint \frac{\mathrm{d}u}{2\pi\mathrm{i}u}\frac{\mathrm{d}v}{2\pi\mathrm{i}v}\{A(z), \chi_i(u)\}\Delta^{-1}_{ij}(u, v)\{\chi_j(v), B(w)\} \tag{5}$$

where $\Delta^{-1}(x, y)$ is the inverse of the Dirac matrix $\Delta_{ij}(x, y) = \{\chi_i(x), \chi_j(y)\}$ in the sense that

$$\oint \frac{\mathrm{d}u}{2\pi\mathrm{i}u}\Delta_{ij}(z, u)\Delta^{-1}_{jk}(u, w) = \delta_{ik} \sum_{n\in\mathbb{Z}} \left(\frac{z}{w}\right)^n = \delta_{ik}\delta\left(\frac{z}{w}\right). \tag{6}$$

Under the Dirac bracket, the remaining current $E^{-}(z)$ with $k = 1$ leads to the Virasoro algebra

$$\{E^{-}(z), E^{-}(w)\}_D = -\mathrm{i}(E^{-}(z) + E^{-}(w)) \sum_{n\in\mathbb{Z}} n \left(\frac{w}{z}\right)^n + \frac{\mathrm{i}}{2} \sum_{n\in\mathbb{Z}} n^3 \left(\frac{w}{z}\right)^n . \tag{7}$$

We now consider the q-deformed Kac–Moody algebra for $sl(2)_q$ of level k defined by [14, 13]

$$[H_n, H_m] = \frac{[2n][kn]}{2n}\delta_{m+n,0} \tag{8}$$

$$[H_0, H_m] = 0 \tag{9}$$

$$[H_n, E^{\pm}_m] = \pm\sqrt{2}q^{\mp|n|\frac{k}{2}}\frac{[2n]}{2n}E^{\pm}_{m+n} \tag{10}$$

$$[H_0, E^{\pm}_m] = \pm\sqrt{2}E^{\pm}_m \tag{11}$$

$$[E^{+}_n, E^{-}_m] = \frac{q^{\frac{k(n-m)}{2}}\Psi_{n+m} - q^{\frac{k(m-n)}{2}}\Phi_{n+m}}{q - q^{-1}} \tag{12}$$

$$E^{\pm}_{n+1}E^{\pm}_m - q^{\pm 2}E^{\pm}_m E^{\pm}_{n+1} = q^{\pm 2}E^{\pm}_n E^{\pm}_{m+1} - E^{\pm}_{m+1}E^{\pm}_n \tag{13}$$

where

$$\Psi(z) = q^{\sqrt{2}H_0}\mathrm{e}^{\sqrt{2}(q-q^{-1})\sum_{n>0}H_n z^{-n}} \tag{14}$$

$$\Phi(z) = q^{-\sqrt{2}H_0}\mathrm{e}^{-\sqrt{2}(q-q^{-1})\sum_{n<0}H_n z^{-n}} \tag{15}$$

and $[x] = \frac{q^x - q^{-x}}{q-q^{-1}}$, leading to the operator product relations

$$H(z)H(w) = \sum_{n>0}\frac{[2n][kn]}{2n}\left(\frac{w}{z}\right)^n \tag{16}$$

$$H(z)E^{\pm}(w) = \pm\sqrt{2}\left(1 + \sum_{n>0}\frac{[2n]}{2n}\left(\frac{wq^{\mp\frac{k}{2}}}{z}\right)^n\right)E^{\pm}(w) \tag{17}$$

$$E^{+}(z)E^{-}(w) = \frac{1}{w(q-q^{-1})}\left(\frac{\Psi(wq^{\frac{k}{2}})}{z-wq^k} - \frac{\Phi(wq^{-\frac{k}{2}})}{z-wq^{-k}}\right) \tag{18}$$

$$E^{\pm}(z)E^{\pm}(w)(z-wq^{\pm 2}) = E^{\pm}(w)E^{\pm}(z)(zq^{\pm 2}-w) \tag{19}$$

for $|z| > |w|$ and we are considering q to be a pure phase. It is clear from (19) that $E^{\pm}$ are not self-commuting objects, however this structure can be constructed using the Wakimoto construction [11]. In particular, for $k = 1$, it can be constructed in terms of a single Fubini Veneziano field [15] as follows

$$E^{\pm}(z) =: \mathrm{e}^{\pm\mathrm{i}\sqrt{2}Q^{\pm}(z)} : \qquad H(z) = \sum_{n\in\mathbb{Z}}\alpha_n z^{-n} \tag{20}$$

where

$$Q^{\pm}(z) = \tilde{q} - \mathrm{i}\tilde{p}\ln z + \mathrm{i}\sum_{n<0}\frac{\alpha_n}{[n]}(zq^{\mp\frac{1}{2}})^{-n} + \mathrm{i}\sum_{n>0}\frac{\alpha_n}{[n]}(zq^{\pm\frac{1}{2}})^{-n} \tag{21}$$

and

$$[\alpha_n, \alpha_m] = \frac{[2n][n]}{2n}\delta_{m+n,0} \qquad [\tilde{q}, \tilde{p}] = \mathrm{i}. \tag{22}$$

We should point out that for $q = 1$, the vertex operator construction (20)–(22) satisfies (1)–(3) with $k = 1$. Our Hamiltonian reduction procedure consists of implementing the following constraints

$$\chi_1^q = \frac{\Psi(z) - \Phi(z)}{\sqrt{2}(q-q^{-1})} \approx 0 \tag{23}$$

$$\chi_2^q = E^{+}(z) \approx 1 \tag{24}$$

for $\Psi(z)$ and $\Phi(z)$ defined in (14) and (15) respectively. Notice that $\chi_1^q = H(z) + \mathrm{O}(q - q^{-1})$, and reduce consistently to the known $q = 1$ case.

For $q \neq 1$, the q-deformed Dirac matrix is constructed out of the following relations obtained by direct calculation using the vertex operators (20)–(22)

$$\Psi(z)\Phi(w) = \frac{(z-wq^3)(z-wq^{-3})}{(z-wq)(z-wq^{-1})}\Phi(w)\Psi(z) \tag{25}$$

$$\Psi(z)E^{\pm}(w) = q^{\pm 2}\frac{(z-wq^{\mp\frac{5}{2}})}{(z-wq^{\pm\frac{3}{2}})}E^{\pm}(w)\Psi(z) \tag{26}$$

$$E^{\pm}(z)\Phi(w) = q^{\pm 2}\frac{(z-wq^{\mp\frac{5}{2}})}{(z-wq^{\pm\frac{3}{2}})}\Phi(w)E^{\pm}(z) \tag{27}$$

(for $|z| > |w|$) together with (18) and (19) for $k = 1$.

From equations (25)–(27) we evaluate

$$\left[\frac{\Psi(z)-\Phi(z)}{\sqrt{2}(q-q^{-1})}, \frac{\Psi(w)-\Phi(w)}{\sqrt{2}(q-q^{-1})}\right] = \frac{[2]}{2}\Phi(w)\Psi(z)\sum_{n>0}\left(\frac{w}{z}\right)^n[n] - \frac{[2]}{2}\Phi(z)\Psi(w)\sum_{n>0}\left(\frac{z}{w}\right)^n[n] \tag{28}$$

$$\left[\frac{\Psi(z)-\Phi(z)}{\sqrt{2}(q-q^{-1})}, E^{\pm}(w)\right] = \pm\frac{[2]}{\sqrt{2}}E^{\pm}(w)\Psi(z)\left(\sum_{n\geqslant 0}\left(\frac{wq^{\pm\frac{3}{2}}}{z}\right)^n - \frac{q^{\mp 1}}{[2]}\right) \pm\frac{[2]}{\sqrt{2}}\Phi(z)E^{\pm}(w)\left(\sum_{n\geqslant 0}\left(\frac{zq^{\pm\frac{3}{2}}}{w}\right)^n - \frac{q^{\mp 1}}{[2]}\right) \tag{29}$$

$$[E^{\pm}(z), E^{\mp}(w)] = \pm\frac{1}{q-q^{-1}}\left(\Psi(wq^{\pm\frac{1}{2}})\sum_{n\in\mathbb{Z}}\left(\frac{wq^{\pm 1}}{z}\right)^n - \Phi(wq^{\mp\frac{1}{2}})\sum_{n\in\mathbb{Z}}\left(\frac{wq^{\mp 1}}{z}\right)^n\right) \tag{30}$$

and

$$[E^{\pm}(z), E^{\pm}(w)] = \pm\frac{1}{2}(q-q^{-1})E^{\pm}(w)E^{\pm}(z)\left([2]\sum_{n\geqslant 0}\left(\frac{wq^{\pm 2}}{z}\right)^n - q^{\mp 1}\right) \mp\frac{1}{2}(q-q^{-1})E^{\pm}(z)E^{\pm}(w)\left([2]\sum_{n\geqslant 0}\left(\frac{zq^{\pm 2}}{w}\right)^n - q^{\mp 1}\right). \tag{31}$$

Notice that the rhs of (28) and (29) is normal ordered and all brackets display explicit antisymmetry under $z \leftrightarrow w$.

Let us now discuss the classical counterpart of the quantum brackets (28)–(31). The usual canonical quantization procedure associates the classical Poisson bracket structure to quantum commutators as

$$\{\,,\,\} \to -\mathrm{i}[\,,\,]. \tag{32}$$

The new feature compared with the $q = 1$ case is the non-vanishing of equationn (31). Moreover, equation (31) presents a quadratic structure which suggests an exponential realization (vertex operator for $k = 1$ or the generalized Wakimoto construction for generic k (see [11])) and the commutators are evaluated using the Baker–Haussdorff formula. The latter has no classical analogue but we still expect a classical counterpart for the quantum algebra (28)–(31) to preserve their structure of algebraic nature.

We propose a classical Poisson bracket algebra by mapping quantum operators $\hat{A}, \hat{B}$ into classical objects A, B such that

$$\{A(z), B(w)\}_{PB} \to \mp\mathrm{i}[\hat{A}(z), \hat{B}(w)] \tag{33}$$

where the plus sign is only taken for $A = B = E^{\pm}$. All other brackets follow the usual correspondence principle (32). Under this prescription and constraints (23) and (24), we construct the Dirac matrix $\Delta_{ij}(z, w) = \{\chi_i(z), \chi_j(w)\}_{PB}$ to be

$$\Delta_{11}(z, w) = -\mathrm{i}\frac{[2]}{2}\left(\sum_{n>0}\left(\frac{w}{z}\right)^n[n] - \sum_{n>0}\left(\frac{z}{w}\right)^n[n]\right) \tag{34}$$

$$\Delta_{12}(z,w) = -\mathrm{i}\frac{[2]}{\sqrt{2}}\left(\sum_{n\geqslant 0}\left(\frac{wq^{\frac{3}{2}}}{z}\right)^n + \sum_{n\geqslant 0}\left(\frac{zq^{\frac{3}{2}}}{w}\right)^n\right) + \frac{2q^{-1}\mathrm{i}}{\sqrt{2}} \tag{35}$$

$$\Delta_{22}(z,w) = \mathrm{i}\frac{[2]}{2}(q-q^{-1})\left(\sum_{n>0}\left(\frac{wq^2}{z}\right)^n - \sum_{n>0}\left(\frac{zq^2}{w}\right)^n\right) \tag{36}$$

and $\Delta_{21}(z,w) = -\Delta_{12}(w,z)$.

Its inverse is defined by equation (6) yielding

$$\Delta^{-1}_{11}(z,w) = \frac{-2\mathrm{i}(q-q^{-1})}{[2]}\left(\sum_{n>0}\left(\frac{w}{z}\right)^n\frac{[n]}{[2n]} - \sum_{n>0}\left(\frac{z}{w}\right)^n\frac{[n]}{[2n]}\right) \tag{37}$$

$$\Delta^{-1}_{12}(z,w) = \frac{-2\mathrm{i}\sqrt{2}}{[2]}\left(\sum_{n>0}\left(\frac{wq^{-\frac{1}{2}}}{z}\right)^n\frac{[n]}{[2n]} + \sum_{n>0}\left(\frac{zq^{-\frac{1}{2}}}{w}\right)^n\frac{[n]}{[2n]}\right) \tag{38}$$

$$\Delta^{-1}_{21}(z,w) = \frac{2\mathrm{i}\sqrt{2}}{[2]}\left(\sum_{n>0}\left(\frac{wq^{-\frac{1}{2}}}{z}\right)^n\frac{[n]}{[2n]} + \sum_{n>0}\left(\frac{zq^{-\frac{1}{2}}}{w}\right)^n\frac{[n]}{[2n]}\right) \tag{39}$$

$$\Delta^{-1}_{22}(z,w) = \frac{2\mathrm{i}}{[2]}\left(\sum_{n>0}\left(\frac{wq^{-2}}{z}\right)^n\frac{[n]^2}{[2n]} - \sum_{n>0}\left(\frac{zq^{-2}}{w}\right)^n\frac{[n]^2}{[2n]}\right) \tag{40}$$

and the Dirac bracket (5) for the remaining current $E^-(z)$ can be evaluated using the modified correspondence principle (33) in equations (28)–(31) yielding, after redefining $\tilde{E}^- = (q-q^{-1})^2\sqrt{\frac{[2]}{2}}E^- + \frac{4}{\sqrt{2}[2]}$

$$\begin{aligned}\{\tilde{E}^-(z),\tilde{E}^-(w)\}_D = {}& \frac{\mathrm{i}[2]}{2}(q-q^{-1})^2\tilde{E}^-(z)\tilde{E}^-(w)\sum_{n\in\mathbb{Z}}q^{-2|n|}\frac{[n]^2}{[2n]}\left(\frac{z}{w}\right)^n \\ & -\mathrm{i}(q-q^{-1})^2\sum_{n\in\mathbb{Z}}q^{-2|n|}[2n]\left(\frac{z}{w}\right)^n.\end{aligned} \tag{41}$$

The algebra given in (41) coincide, apart from the factor $q^{-2|n|}$ to the q-Virasoro algebra proposed by Frenkel and Reshetikhin (see [5]) with $q = \mathrm{e}^{\mathrm{i}h}$. This undesirable factor may be absorbed by redefining the classical brackets (34)–(36) of the form

$$\{A(z), B(w)\} = \sum_{n\in\mathbb{Z}}C_n\left(\frac{z}{w}\right)^n$$

into

$$\{A(z), B(w)\} = \sum_{n\in\mathbb{Z}}C_n\left(\frac{z}{w}\right)^n q^{2|n|}.$$

Under this modification of the correspondence principle (33) the Dirac bracket for the remaining current $\tilde{E}^-(z)$ coincided precisely with the algebra given in [5], namely,

$$\begin{aligned}\{\tilde{E}^-(z),\tilde{E}^-(w)\}_D = {}& \frac{\mathrm{i}[2]}{2}(q-q^{-1})^2\tilde{E}^-(z)\tilde{E}^-(w)\sum_{n\in\mathbb{Z}}\frac{[n]^2}{[2n]}\left(\frac{z}{w}\right)^n \\ & -\mathrm{i}(q-q^{-1})^2\sum_{n\in\mathbb{Z}}[2n]\left(\frac{z}{w}\right)^n.\end{aligned} \tag{42}$$

The differing factor $q^{2|n|}$ is viewed of quantum origin. It is known, for instance, in quantizing the $SU(2)$ WZW model, that the coupling constant of the diagonal fields is shifted by a factor of 2 (Coxeter number of $SU(2)$ (see [2, 3]). In general we expect the quantum correction associated to a q-deformed Kac–Moody algebra $\hat{g}$ to be given as

$$\{A(z), B(w)\} = \sum_{n\in\mathbb{Z}} C_n \left(\frac{z}{w}\right)^n q^{h|n|}$$

where h is the Coxeter element of g. For the general q-deformed Kac–Moody algebra $\hat{g}$, if we follow the usual constraints connecting the g invariant WZW and the conformal Toda models [9] we obtain the q-deformed W_n-algebra by adding to the q-Virasoro (42), generators of higher spin. The construction of a classical action invariant under transformations generated by operators satisfying the proposed classical Poisson algebra is also an interesting problem that is under investigation and shall be reported in a future publication.

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